1 Linear maps and the Rank theorem

1.1 Linear maps

Definition 1. Let $f\colon S \longrightarrow S'$ be a function. The **image or range** of f, denoted $\mathrm{Im}(f)$, is a subset of S' given by:

$$\mathrm{Im}(f) = \{y \in S' \mid \text{ there exist } x \in S \text{ such that } f(x) = y\}.$$

Definition 2. Suppose that V and V' are both **vector spaces** over $\mathbb{R}$. A function $f\colon V \longrightarrow V'$ is a called a **linear map** if it satisfies:

1. $f(x + x') = f(x) + f(x')$ for all $x, x' \in V$.
2. $f(ax) = af(x)$ for all $x \in V$ and $a \in \mathbb{R}$

Example 3. Take the vector space $C^1(\mathbb{R})$ of continuously differentiable functions on $\mathbb{R}$. And consider the derivative $D\colon C^1(\mathbb{R}) \longrightarrow C^1(\mathbb{R})$. This is an example of a linear map on a vector space of infinite dimension since:

$$D(f+g) = D(f) + D(g) \qquad \text{and} \qquad D(af) = aD(f) \qquad \text{for all} \quad a \in \mathbb{R},\ f \in C^1(\mathbb{R}).$$

The same map D also defines a linear map on the vector space P_n of polynomial functions of degree at most n.

Example 4. Take $V = \mathbb{R}^n$, $V' = \mathbb{R}^m$ and $f\colon V \longrightarrow V'$ given by $x \mapsto Ax$ where A is a $m \times n$-matrix. This is, for $x \in \mathbb{R}^n$, the result $y = f(x)$ is in $\mathbb{R}^m$:

$$f(x) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

where $y_i = a_{i,1}x_1 + \dots a_{i,n}x_n$ for each $i = 1, \dots, m$.

Proposition 5. *A map $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear if and only if it is given as $f(x) = Ax$ for some $m \times n$-matrix A.*

Proof. The linear map f is determined by the values on the basis $\{e_1, e_2, \dots, e_n\}$ of $\mathbb{R}^n$. On the other hand, the evaluation at each of this vectors is uniquely written as linear combination of the basis $\{e'_1, \dots, e'_m\}$ in $\mathbb{R}^m$. We can find therefore, for any linear map $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ numbers $a_{i,j}$ such that

$$f(e_j) = \sum_{i=1}^{m} a_{ij} e'_i,$$

providing us with a matrix representation for f given by the matrix $A = (a_{i,j})$. $\square$

Once we are working with vector spaces and linear maps, we can define:

Definition 6. Suppose that V and V' are both vector spaces over $\mathbb{R}$ and $f\colon V \longrightarrow V'$ is a linear map. The kernel of f, denoted $\ker(f)$, is defined to be

$$\ker(f) = \{x \in V \mid f(x) = 0\}$$

Proposition 7. *Suppose that V and V' are both vector spaces over $\mathbb{R}$ and $f\colon V \longrightarrow V'$ is a linear map. Then $\ker(f)$ is a subspace of V and the image Im(f) is a subspace of V'.*

Definition 8. Suppose that V and V' are both vector spaces over $\mathbb{R}$ and $f\colon V \longrightarrow V'$ is a linear map. The dimension of $\ker(f)$ is called the **nullity** of f. The dimension of $\mathrm{Im}(f)$ is called the **rank** of f.

Proposition 9. *Let $f(x) = Ax$ be a linear map $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, then the rank and the nullity of f coincide with the rank and the nullity of A.*

Proof. This is not hard to see from the definitions of rank and nullity. The dimension of column space is the dimension of the image

$$\mathrm{Im}(f) = x_1c_1 + \dots x_nc_n.$$

On the other hand, the NullSpace of A is just the kernel of the map $f(x) = A(x)$. $\square$

Theorem 10. ***(Rank Theorem)*** *Suppose that V and V' are both vector spaces over $\mathbb{R}$ and $f\colon V \longrightarrow V'$ is a linear map. Assume that $\dim(V) = n$ is finite, then we have the equality:*

$$\textit{nullity}(f) + \textit{rank}(f) = n.$$

Proof. Let $k = \mathrm{nullity}(f)$ and consider a basis $\{v_1, \dots, v_k\}$ of $\ker(f)$ as a subspace of V. We can always extend this basis, to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of the whole V. We would like to prove that the system of vectors $\{f(v_{k+1}), \dots, f(v_n)\}$ is a basis of the subspace $\mathrm{Im}(f)$ of V'.

1. (Span) Let $y \in \mathrm{Im}(f)$. Then $y = f(x)$, for some $x \in V$. As x is an element of V and $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ is a basis, we can find a unique set of coefficients $a_i \in \mathbb{R}$ such that $x = a_1v_1 + \cdots + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n$. But then:

$$\begin{aligned} y = f(x) &= f(a_1v_1 + \cdots + a_kv_k + a_{k+1}v_{k+1} + \cdots + a_nv_n) \\ &= a_1f(v_1) + \cdots + a_kf(v_k) + a_{k+1}f(v_{k+1}) + \cdots + a_nf(v_n) \\ &= a_1 \cdot (0) + \cdots + a_k \cdot (0) + a_{k+1}f(v_{k+1}) + \cdots + a_nf(v_n) \\ &= a_{k+1}f(v_{k+1}) + \cdots + a_nf(v_n). \end{aligned}$$

   And we obtain that the system of vectors $\{f(v_{k+1}), \dots, f(v_n)\}$ span the image $\mathrm{Im}(f)$.

2. (Linearly independent) Suppose that a linear combination of $\{f(v_{k+1}), \ldots, f(v_n)\}$ gives 0. Say that we have

$$\lambda_{k+1} f(v_{k+1}) + \cdots + \lambda_n f(v_n) = 0.$$

But then $f(\lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n) = \lambda_{k+1} f(v_{k+1}) + \cdots + \lambda_n f(v_n) = 0$ and the element $x' = \lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n$ will be in the kernel of f. Since $\{v_1, \ldots, v_k\}$ is a basis of $\ker(f)$ and the whole system $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ is linearly independent, the only possibility is that $x' = 0$ and

$$\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0.$$

We have therefore obtained that the vectors $\{f(v_{k+1}, \ldots, f(v_n)\}$ are linearly independent in V'.

As a combination of (1) and (2), we have proven that $\{f(v_{k+1}), \ldots, f(v_n)\}$ is a basis of $\mathrm{Im}(f)$ and the dimension of $\mathrm{Im}(f)$, which is the rank of f, is $n - k$. In any case, we have:

$$\mathrm{nullity}(f) + \mathrm{rank}(f) = k + (n - k) = n.$$

□

Definition 11. Let $f \colon S \longrightarrow S'$ be a function between sets S and S'. The function f is said to be:

(1) **injective or one-to-one** if $f(x) = f(x') \Rightarrow x = x'$ and

(2) **surjective or onto** if $\mathrm{Im}(f) = S$.

A function that is both injective and surjective is called **bijective**.

Proposition 12. *Suppose that V and V' are both vector spaces over $\mathbb{R}$ and $f \colon V \longrightarrow V'$ is a linear map.*

(1) The map f is injective if and only if the kernel is trivial subspace $\ker(f) = \{0\}$.

(2) Assuming V' has finite dimension, the map f is surjective if and only if the rank rank$(f) = \dim(V')$.

Proof. (1) The condition $\ker(f) = \{0\}$ is clearly necessary. On the hard

$$f(x) = f(x') \Rightarrow f(x - x') = 0 \Rightarrow x - x' \in \ker(f).$$

Therefore the condition $\ker(f) = \{0\}$ is sufficient as well.
(2) The subspace $\mathrm{Im}(f)$ has dimension equal to $\dim(V')$ if and only if $V' = \mathrm{Im}(f)$. □

Corollary 13. *If $f \colon V \longrightarrow V'$ is a bijective linear map between finite dimension vector spaces V and V', then $\dim(V) = \dim(V')$.*